NSW Education Standards Authority

2018 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions	• Reading time – 5 minutes • Working time – 3 hours • Write using black pen • Calculators approved by NESA may be used • A reference sheet is provided at the back of this paper • In Questions 11–16, show relevant mathematical reasoning and/or calculations
Total marks: 100	**Section I – 10 marks** (pages 2–7) • Attempt Questions 1–10 • Allow about 15 minutes for this section **Section II – 90 marks** (pages 8–19) • Attempt Questions 11–16 • Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which expression is equal to $\int \frac{1}{\sqrt{1-4x^2}}\,dx$?

A. $\frac{1}{2}\sin^{-1}\frac{x}{2} + C$

B. $\frac{1}{2}\sin^{-1}2x + C$

C. $\sin^{-1}\frac{x}{2} + C$

D. $\sin^{-1}2x + C$

2 What are the equations of the asymptotes of the hyperbola $9x^2 - 4y^2 = 36$?

A. $y = \pm\frac{9}{4}x$

B. $y = \pm\frac{2}{3}x$

C. $y = \pm\frac{3}{2}x$

D. $y = \pm\frac{4}{9}x$

3 The cubic equation $x^3 + 2x^2 + 5x - 1 = 0$ has roots α, β and γ.

Which cubic equation has roots $\frac{-1}{\alpha}, \frac{-1}{\beta}, \frac{-1}{\gamma}$?

A. $x^3 - 5x^2 - 2x + 1 = 0$

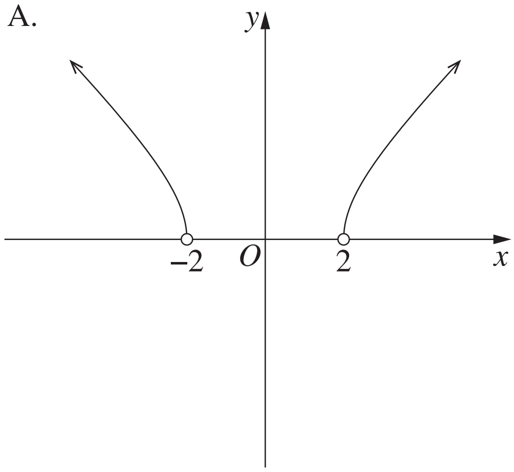
B. $x^3 - 5x^2 - 2x - 1 = 0$

C. $x^3 + 5x^2 + 2x + 1 = 0$

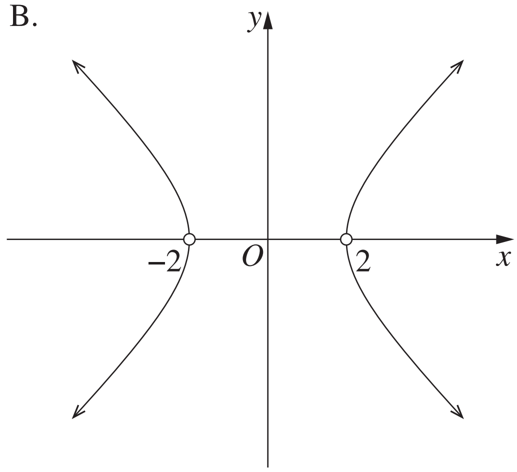
D. $x^3 + 5x^2 - 2x + 1 = 0$

4 Which graph best represents the curve $y = \dfrac{1}{\sqrt{x^2 - 4}}$?

A.



B.



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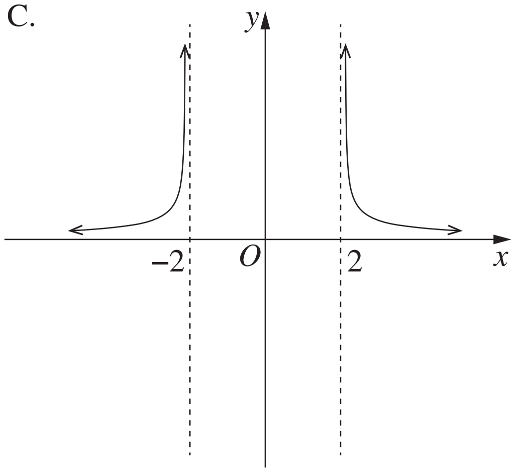
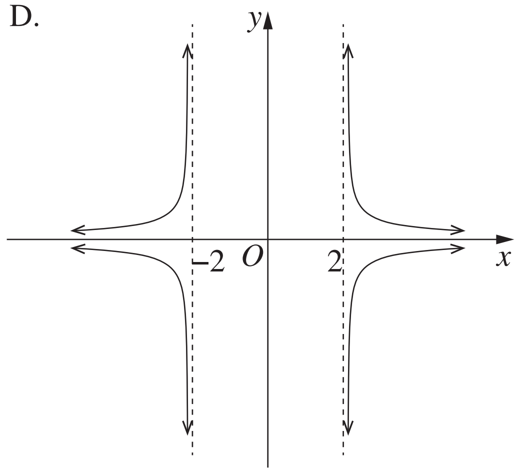
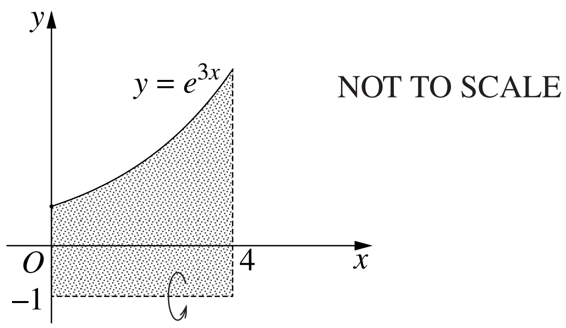


D.



5 The diagram shows the graph $y = e^{3x}$ for $0 \le x \le 4$. The region bounded by $y = -1$, $y = e^{3x}$, $x = 0$ and $x = 4$ is rotated about the line $y = -1$ to form a solid.



Which integral represents the volume of the solid formed?

A. $\pi \int_0^4 \left(e^{3x} + 1\right)^2 dx$

B. $2\pi \int_0^4 x\left(e^{3x} + 1\right) dx$

C. $\pi \int_0^4 \left(e^{3x} - 1\right)^2 dx$

D. $2\pi \int_0^4 x\left(e^{3x} - 1\right) dx$

6 Which complex number is a 6th root of i?

A. $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

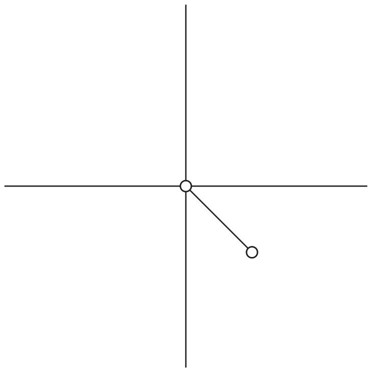
B. $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

C. $-\sqrt{2} + \sqrt{2} i$

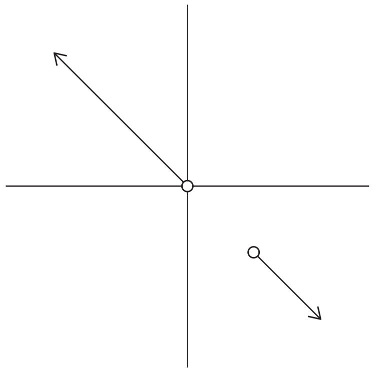
D. $-\sqrt{2} - \sqrt{2} i$

7 Which diagram best represents the solutions to the equation $\arg(z) = \arg(z + 1 - i)$?

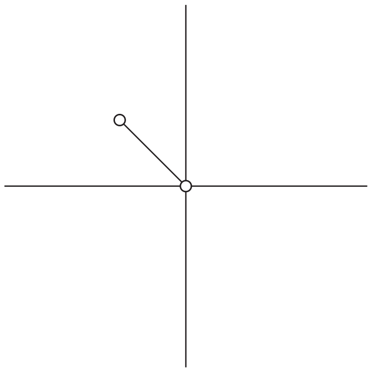
A.



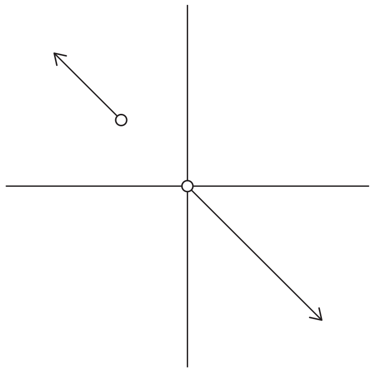
B.



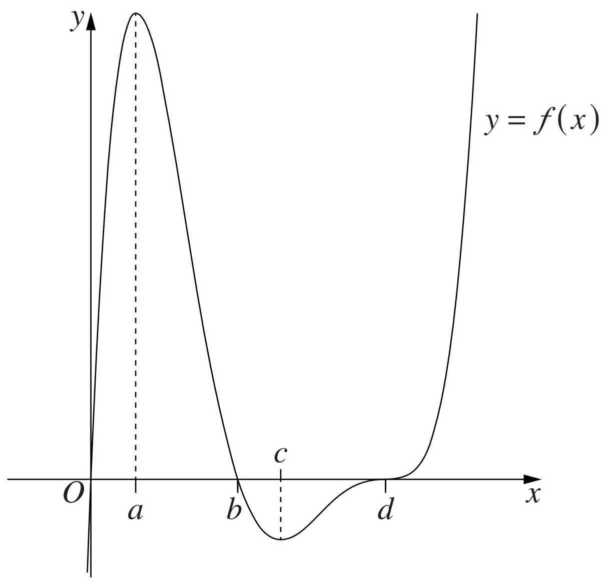
C.



D.



8 The diagram shows the graph of the curve $y = f(x)$.



Let $F(x) = \int_0^x f(t)\,dt$.

At what value(s) of x does the concavity of the curve $y = F(x)$ change?

A. d

B. a, c

C. b, d

D. a, c, d

9 It is given that a, b are real and p, q are purely imaginary.

Which pair of inequalities must always be true?

A. $a^2 p^2 + b^2 q^2 \le 2abpq, \quad a^2 b^2 + p^2 q^2 \le 2abpq$

B. $a^2 p^2 + b^2 q^2 \le 2abpq, \quad a^2 b^2 + p^2 q^2 \ge 2abpq$

C. $a^2 p^2 + b^2 q^2 \ge 2abpq, \quad a^2 b^2 + p^2 q^2 \le 2abpq$

D. $a^2 p^2 + b^2 q^2 \ge 2abpq, \quad a^2 b^2 + p^2 q^2 \ge 2abpq$

10 Consider the functions $f(x) = \sin x$ and $g(x) = x \sin x$.

The x-coordinate of each stationary point of $f(x)$ is very close to the x-coordinate of a stationary point of $g(x)$.

Suppose $f(x)$ has a stationary point at $x = a$ and the stationary point of $g(x)$ with x-coordinate closest to $x = a$ is at $x = b$.

Which statement is always true?

A. $a < b$

B. $a > b$

C. $|a| < |b|$

D. $|a| > |b|$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Let $z = 2 + 3i$ and $w = 1 - i$.

(i) Find zw. **1**

(ii) Express $\overline{z} - \dfrac{2}{w}$ in the form $x + iy$, where x and y are real numbers. **2**

(b) The polynomial $p(x) = x^3 + ax^2 + b$ has a zero at r and a double zero at 4. **3**

Find the values of a, b and r.

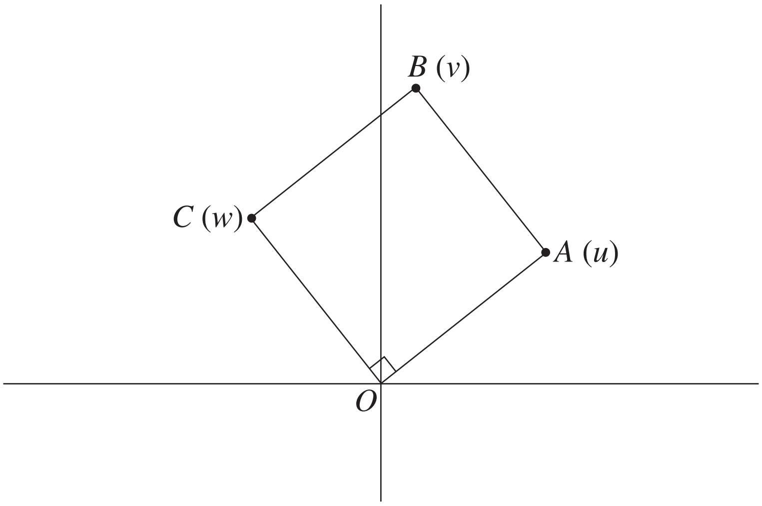
(c) By writing $\dfrac{x^2 - x - 6}{(x+1)(x^2-3)}$ in the form $\dfrac{a}{x+1} + \dfrac{bx+c}{x^2-3}$, **4**

find $\displaystyle\int \frac{x^2 - x - 6}{(x+1)(x^2-3)}\,dx$.

Question 11 continues on page 9

(d) The points A, B and C on the Argand diagram represent the complex numbers u, v and w respectively.

The points O, A, B and C form a square as shown on the diagram.



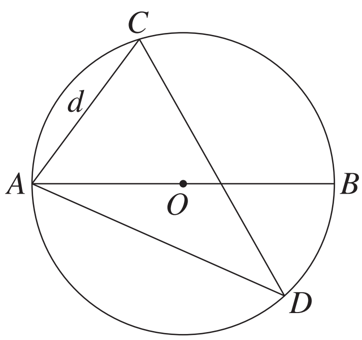
It is given that $u = 5 + 2i$.

(i) Find w. **1**

(ii) Find v. **1**

(iii) Find $\arg\left(\dfrac{w}{v}\right)$. **1**

(e) The circle centred at O with radius r has a diameter AB. Points C and D are chosen on the circle as shown in the diagram. The chord AC has length d. **2**

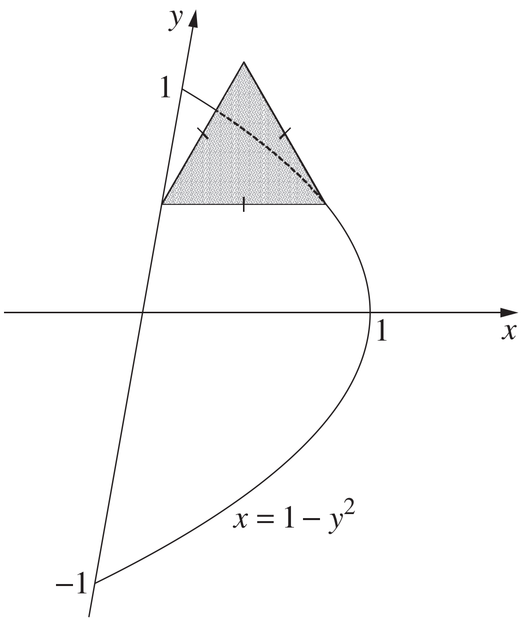


Show that $d = 2r \sin D$.

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) The base of a solid is the region enclosed by the parabola $x = 1 - y^2$ and the y-axis. Each cross-section perpendicular to the y-axis is an equilateral triangle, as shown in the diagram. **3**



Find the volume of the solid.

(b) A curve has equation $x^2 + xy + y^2 = 3$.

(i) Use implicit differentiation to show that $\dfrac{dy}{dx} = -\dfrac{2x + y}{x + 2y}$. **2**

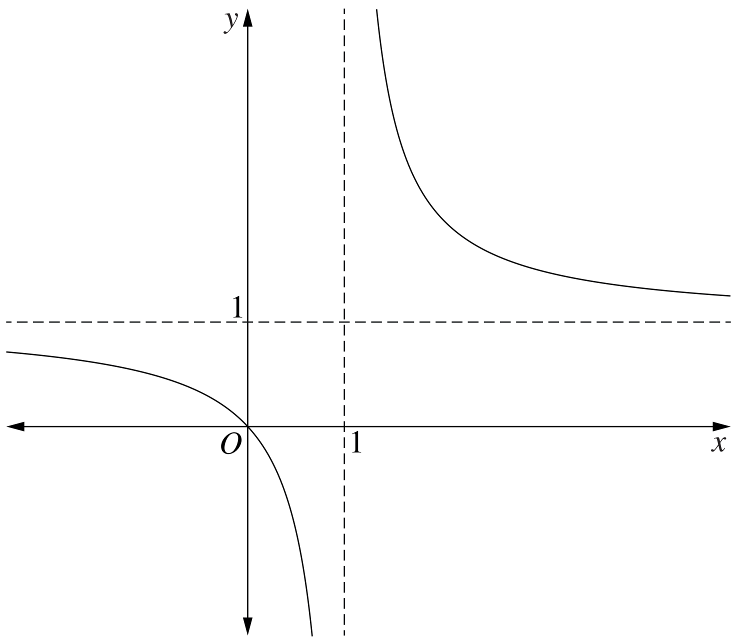
(ii) Hence, or otherwise, find the coordinates of the points on the curve where $\dfrac{dy}{dx} = 0$. **2**

(c) Find $\displaystyle\int \frac{x^2 + 2x}{x^2 + 2x + 5}\,dx$. **3**

Question 12 continues on page 11

Question 12 (continued)

(d) The diagram shows the graph of the function $f(x) = \dfrac{x}{x-1}$.



Draw a separate half-page graph for each of the following functions, showing all asymptotes and intercepts.

(i) $y = |f(x)|$ **1**

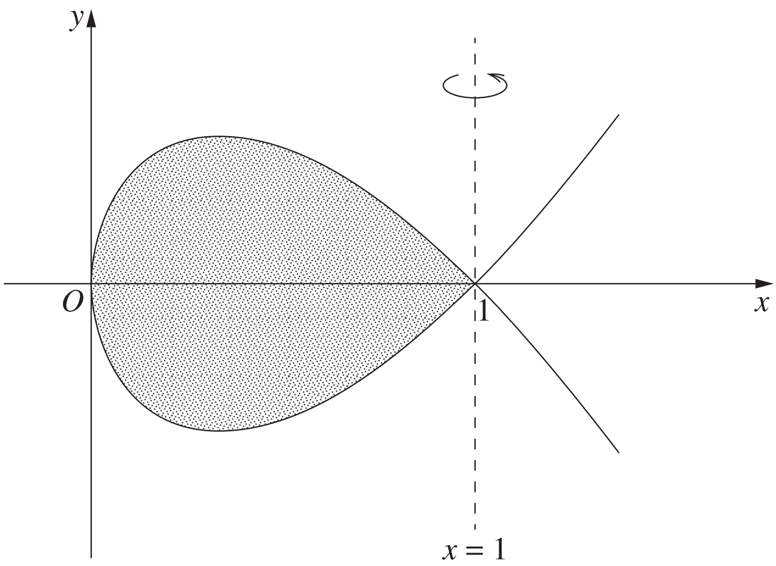
(ii) $y = (f(x))^2$ **2**

(iii) $y = x + f(x)$ **2**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) The graph $y^2 = x(1-x)^2$ is shown. **3**



Use the method of cylindrical shells to find the volume of the solid formed when the shaded region is rotated about the line $x = 1$.

(b) Let $z = 1 - \cos 2\theta + i\sin 2\theta$, where $0 < \theta \leq \pi$.

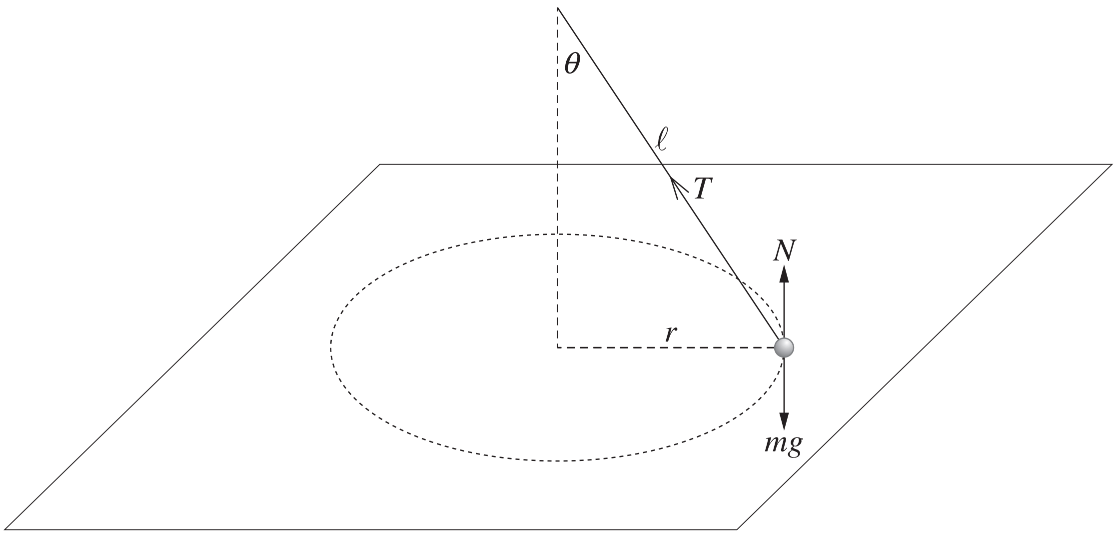
(i) Show that $|z| = 2\sin\theta$. **2**

(ii) Show that $\arg(z) = \dfrac{\pi}{2} - \theta$. **2**

Question 13 continues on page 13

(c) A particle of mass m is attached to a light inextensible string of length ℓ. The string makes an angle θ to the vertical. **3**

The particle is moving in a circle of radius r on a smooth horizontal surface. The particle is moving with uniform angular velocity ω. The forces on the particle are the tension T in the string; the normal reaction N to the horizontal surface; and the gravitational force mg.



The particle remains in contact with the horizontal surface.

By resolving the forces on the particle in the horizontal and vertical directions, show that

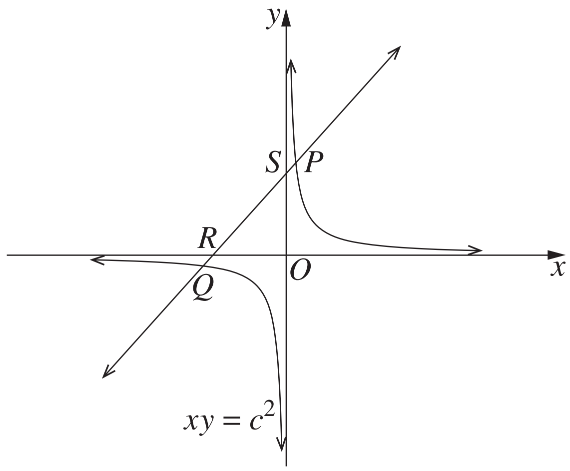
$$\omega^2 \leq \frac{g}{\ell \cos\theta}.$$

Question 13 continues on page 14

Question 13 (continued)

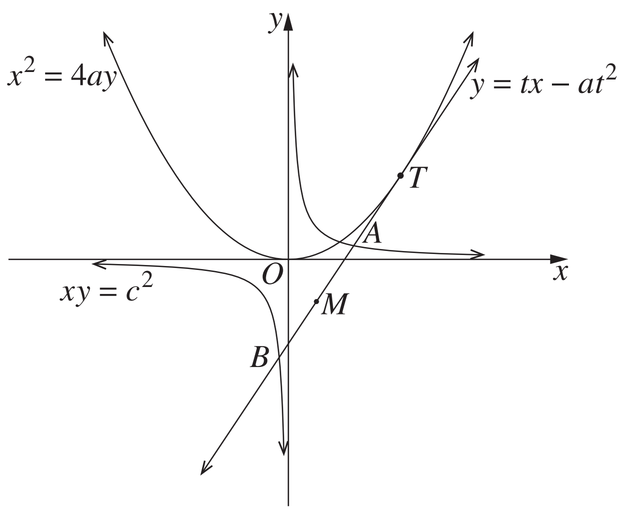
(d) The points $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$ lie on the rectangular hyperbola $xy = c^2$. The line PQ has equation $x + pqy = c(p + q)$. (Do NOT prove this.)

The x and y intercepts of PQ are R and S respectively, as shown in the diagram.



(i) Show that $PS = QR$. **3**

The point $T\left(2at, at^2\right)$ lies on the parabola $x^2 = 4ay$. The tangent to the parabola at T intersects the rectangular hyperbola $xy = c^2$ at A and B and has equation $y = tx - at^2$. (Do NOT prove this.) The point M is the midpoint of the interval AB. One such case is shown in the diagram.



(ii) Using part (i), or otherwise, show that M lies on the parabola $2x^2 = -ay$. **2**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) Using the substitution $t = \tan\frac{\theta}{2}$ evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2 - \cos\theta}$. **3**

(b) A falling particle experiences forces due to gravity and air resistance. The acceleration of the particle is $g - kv^2$, where g and k are positive constants and v is the speed of the particle. (Do NOT prove this.) **3**

Prove that, after falling from rest through a distance, h, the speed of the particle will be $\sqrt{\frac{g}{k}\left(1 - e^{-2kh}\right)}$.

(c) Let $I_n = \int_{-3}^{0} x^n \sqrt{x+3}\, dx$ for $n = 0, 1, 2, \ldots$

(i) Show that, for $n \geq 1$, **3**

$$I_n = \frac{-6n}{3 + 2n} I_{n-1}.$$

(ii) Find the value of I_2. **2**

(d) Three people, A, B and C, play a series of n games, where $n \geq 2$. In each of the games there is one winner and each of the players is equally likely to win.

(i) What is the probability that player A wins every game? **1**

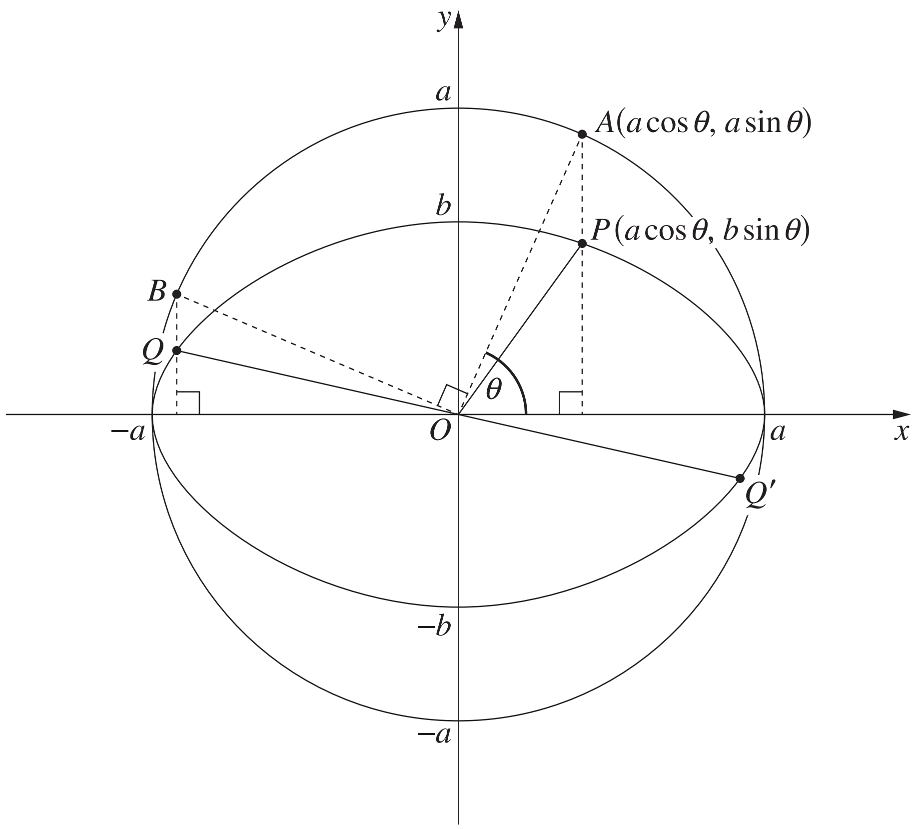
(ii) Show that the probability that A and B win at least one game each but C never wins, is $\left(\frac{2}{3}\right)^n - 2\left(\frac{1}{3}\right)^n$. **1**

(iii) Show that the probability that each player wins at least one game is **2**

$$\frac{3^{n-1} - 2^n + 1}{3^{n-1}}.$$

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) The point $P(a\cos\theta, b\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$, lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$. The point $A(a\cos\theta, a\sin\theta)$ lies vertically above P on the auxiliary circle $x^2 + y^2 = a^2$. The point B lies on the auxiliary circle such that $\angle AOB = \frac{\pi}{2}$ and the point Q lies on the ellipse vertically below B, as shown.



(i) Show that Q has coordinates $(-a\sin\theta, b\cos\theta)$. **2**

The line QO meets the ellipse again at $Q'(a\sin\theta, -b\cos\theta)$. (Do NOT prove this.)

(ii) Show that the minimum size of $\angle POQ'$ is $\tan^{-1}\left(\frac{2ab}{a^2 - b^2}\right)$. **3**

Question 15 continues on page 17

(b) (i) Use De Moivre's theorem and the expansion of $(\cos\theta + i\sin\theta)^8$ to show that **2**

$$\sin 8\theta = \binom{8}{1}\cos^7\theta\sin\theta - \binom{8}{3}\cos^5\theta\sin^3\theta + \binom{8}{5}\cos^3\theta\sin^5\theta - \binom{8}{7}\cos\theta\sin^7\theta.$$

(ii) Hence, show that **3**

$$\frac{\sin 8\theta}{\sin 2\theta} = 4\left(1 - 10\sin^2\theta + 24\sin^4\theta - 16\sin^6\theta\right).$$

(c) Let n be a positive integer and let x be a positive real number.

(i) Show that $x^n - 1 - n\left(x - 1\right) = \left(x - 1\right)\left(1 + x + x^2 + \cdots + x^{n-1} - n\right)$. **1**

(ii) Hence show that $x^n \geq 1 + n(x - 1)$. **2**

(iii) Deduce that for positive real numbers a and b, **2**

$$a^n b^{1-n} \geq na + (1 - n)b.$$

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet.

(a) Use mathematical induction to prove that, for $n \geq 1$, **3**

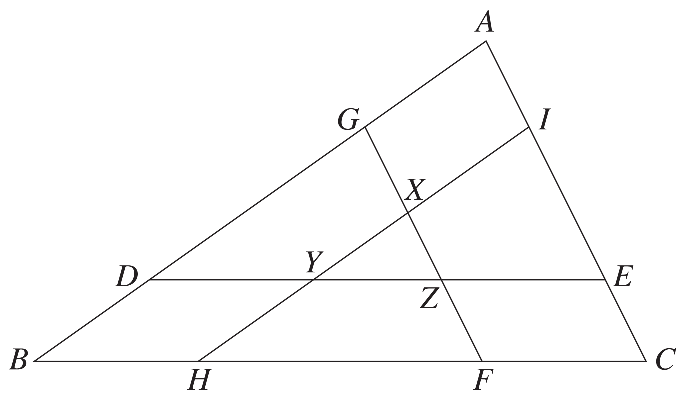
$$x^{\left(3^n\right)} - 1 = \left(x - 1\right)\left(x^2 + x + 1\right)\left(x^6 + x^3 + 1\right)\cdots\left(x^{\left(2\times 3^{n-1}\right)} + x^{\left(3^{n-1}\right)} + 1\right).$$

(b) In $\triangle ABC$, point D is chosen on side AB and point E is chosen on side AC so that DE is parallel to BC and $\dfrac{BC}{DE} = \sqrt{2}$.

The process is repeated two more times. Point F is chosen on BC and point G on BA so that FG is parallel to CA and $\dfrac{CA}{FG} = \sqrt{2}$.

Point H is chosen on side CB and I on CA so that HI is parallel to BA and $\dfrac{BA}{HI} = \sqrt{2}$.

The segments FG and HI intersect at X, DE and HI intersect at Y, and DE and FG intersect at Z.



(i) Prove that $DY = ZE$. **3**

(ii) Find the exact value of the ratio $\dfrac{YZ}{BC}$. **2**

Question 16 continues on page 19

Question 16 (continued)

(c) Let α, β, γ be the zeros of $p(x) = x^3 + px + q$, where p and q are real and $q \neq 0$.

(i) Show that $(\beta - \gamma)^2 = \alpha^2 + \dfrac{4q}{\alpha}$. **2**

(ii) By considering the constant term of a cubic equation with roots $(\alpha - \beta)^2$, $(\beta - \gamma)^2$ and $(\gamma - \alpha)^2$, or otherwise, show that **3**

$$(\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2 = -\left(27q^2 + 4p^3\right).$$

(iii) Deduce that if $27q^2 + 4p^3 < 0$, then the equation $p(x) = 0$ has 3 distinct real roots. **2**

End of paper

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NSW Education Standards Authority

2018 HIGHER SCHOOL CERTIFICATE EXAMINATION

REFERENCE SHEET

– Mathematics –

– Mathematics Extension 1 –

– Mathematics Extension 2 –

Mathematics

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$

$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

Angle sum of a polygon

$$S = (n-2) \times 180^\circ$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

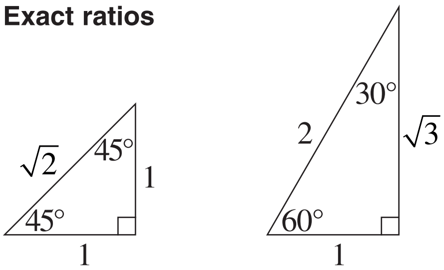
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{\left|ax_1 + by_1 + c\right|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point–gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1} \quad \text{or} \quad S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1 - r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\frac{dy}{dx} = F'(u)\frac{du}{dx}$

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180° = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$$\text{Area} = \frac{1}{2}r^2\theta$$

Mathematics Extension 1

Angle sum identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

t formulae

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta = a, \qquad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \qquad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \qquad \theta = n\pi + \tan^{-1} a$

Division of an interval in a given ratio

$$\left(\frac{mx_2+nx_1}{m+n},\ \frac{my_2+ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $\left(2at,\ at^2\right)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At $\left(x_1,\ y_1\right)$,

tangent: $xx_1 = 2a\left(y + y_1\right)$

normal: $y - y_1 = -\frac{2a}{x_1}\left(x - x_1\right)$

Chord of contact from $\left(x_0,\ y_0\right)$: $xx_0 = 2a\left(y + y_0\right)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$$x = b + a\cos\left(nt+\alpha\right)$$

$$\ddot{x} = -n^2\left(x-b\right)$$

Further integrals

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha+\beta+\gamma = -\frac{b}{a}$$

$$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f\left(x_1\right)}{f'\left(x_1\right)}$$

Binomial theorem

$$\left(a+b\right)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$